Gromov compactness:

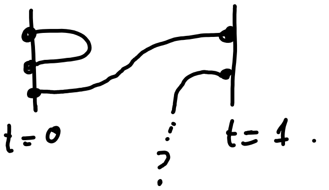
Goal = would like to define invariants that count J-hol. curves.

For this, need to understand compactness / compactification of moduli spaces, so that

→ counts of 0-dim^{l} $\mathcal{M}$ (or $\int$ of cohomology classes on $\mathcal{M}$) are well-def^{d} numbers

→ invariance property: need $\bigsqcup_{t\in[0,1]} \mathcal{M}_t$ compact cobordism between $\mathcal{M}_0$ & $\mathcal{M}_1$?

(so $\#\mathcal{M}(J)$ same for all $J\in\mathcal{J}_{reg}$?)



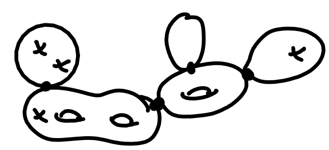
→ $\partial^2 = 0$ etc. on Lagr HF will involve boundary strata of compactified 1-dim. $\mathcal{M}$'s.

This is where we need the symplectic form.

Thm: (Gromov compactness)

$u_n : \Sigma_n \to M$ sequence of J-holom. curves, $J\in\mathcal{J}(M,\omega)$,
$E(u_n) = \int_{\Sigma_n} u_n^*\omega = \langle[\omega], u_{n*}[\Sigma_n]\rangle$ bounded $\Rightarrow$
$\exists$ subsequence that converges to a stable map $u_\infty : \Sigma_\infty \to M$

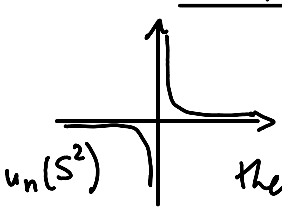
ie: $\Sigma_\infty = \bigcup$ nodal Riemann surfaces



all marked points & nodes are distinct in the domain
(if they come together, create a constant bubble to keep them separated)

Phenomenon: besides possible degeneration of domain (Σ_n, j_n) to a nodal curve,
the main phenomenon is bubbling of spheres (& of discs when $\partial\Sigma \neq \phi$)

Example: $u_n : S^2 = \mathbb{CP}^1 = \mathbb{C}\cup\{\infty\} \longrightarrow \mathbb{CP}^1\times\mathbb{CP}^1$
$(x_0 : x_1) \longmapsto (x_0 : x_1), (n x_1 : x_0)$

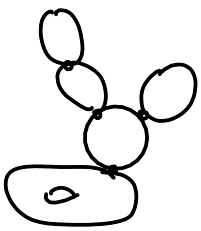


(in affine chart $x = x_1/x_0$: $x \longmapsto \left(x, \frac{1}{nx}\right)$ + extend at 0 & ∞)

then away from origin, uniform convergence to $x \longmapsto (x, 0)$
so limit seems to be just 1st coord. axis -- missing part!
but if we reparametrize : $\tilde{x} = nx$, then get $\tilde{x} \longmapsto \left(\frac{\tilde{x}}{n}, \frac{1}{\tilde{x}}\right)$
uniform cv away from ∞ to $\tilde{x} \mapsto \left(0, \frac{1}{\tilde{x}}\right)$ → 2nd coord axis ✓
Actual domain of limit curve is $\mathbb{CP}^1 \vee \mathbb{CP}^1$ (identify $x = 0$ & $\tilde{x} = \infty$).

Ideas: • domain degenerations: stable compactification $\overline{\mathcal{M}}_{g,k} = \{$stable genus g curves & marked points$\}$ of $\mathcal{M}_{g,k} = \{(\Sigma, j)\}$
⇒ gives a candidate limit domain
when (Σ_n, j_n) don't converge in $\mathcal{M}_{g,k}$.

- identify bubbling regions = where $\sup |du_n| \to \infty$
  Outside of these, Arzela-Ascoli gives a convergent subsequence.
- in bubbling regions, rescale domain: $v_n(z) := u_n(z^0_n + \varepsilon_n z)$,
  $\varepsilon_n \to 0$ suitably chosen $\Rightarrow$ a subsequence of v_n converges to
  a map $v_\infty : \mathbb{C} \to M$, which by removable sing. theorem extends
  to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$: the bubble!
  (when bubbling is at boundary of Σ, limit only def. on half-plane $v_\infty : \mathbb{H} \to (M,L)$
  and removable sing. theorem gives extension to $\mathbb{H} \cup \{\infty\} \simeq D^2$; disc bubble).



- intermediate bubbling stages $\Rightarrow$ might need various rescalings to
  catch all bubbles. ("deepest bubble" $\varepsilon_n = \sup |du_n|^{-1}$ ...).
- The process is <u>finite</u> because of energy estimates:

$$E = \int u^*\omega \geq \hbar > 0$$ for all nonconstant closed J-hol. curves (or J-hol. curves w/ boundary on L)

$\hbar$: minimum energy (by monotonicity lemma, see HW 3, & curvature + inj. radius bounds)

and we've assumed an upper bound on total energy.

- Domain degenerations: <u>stable curves</u> $= \left(\bigcup \text{Riem. surfaces with marked points (all distinct)}\right)/\sim$ identify pairs of marked points at <u>nodes</u>

st. each component is stable (Aut discrete). (sphere have ≥ 3 pts, tori have ≥ 1 pt)

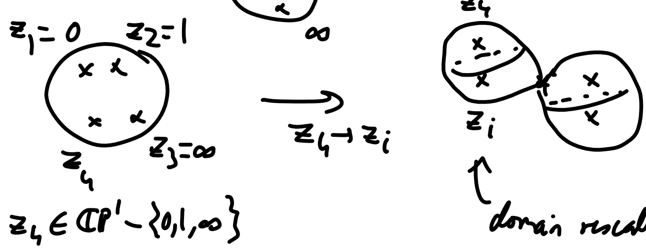
<u>Closed case</u>: $\overline{\mathcal{M}}_{g,k} = \left\{ \begin{array}{l} \text{stable curves of genus } g \ (= \Sigma \text{ genus of components} + \#\text{ loops in gluing graph}) \\ k \text{ marked points (besides the pairs corresp. to the nodes).} \end{array} \right\} / \text{isom}$

Deligne-Mumford moduli space

eg $\overline{\mathcal{M}}_{0,k}$ = trees of spheres, each with at least 3 special pts (marked pts + nodes).

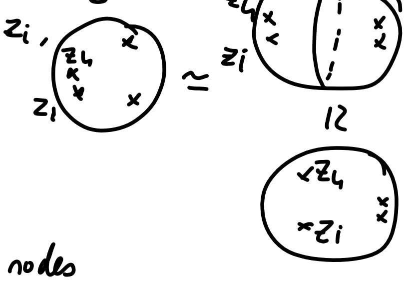
$\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \{pt\}$

$\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1$:

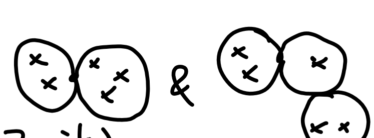


$z_4 \in \mathbb{CP}^1 - \{0,1,\infty\}$

domain rescaling: as $z_4 \to z_i$,

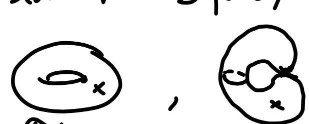


$\overline{\mathcal{M}}_{0,5}$ has things like



($\simeq \mathbb{P}^1 \times \mathbb{P}^1$ blown up at 3 points)

$\overline{\mathcal{M}}_{1,1} \ni$

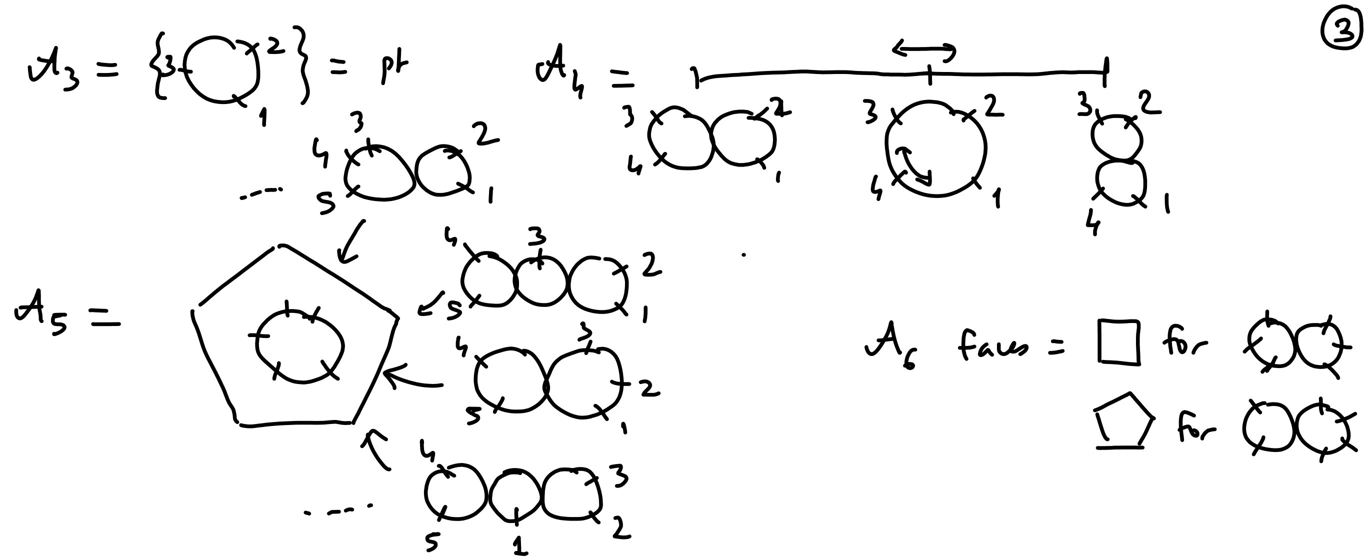


$\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

<u>NB: $\text{codim}_{\mathbb{C}}$ = # nodes</u>

<u>Disc case</u>: $\mathcal{A}_k$ = {stable discs w/ k boundary marked pts (in order along $\partial\Sigma$)} = <u>associahedron</u> (Stasheff) = polytope of dim. $k-3$.
Strata have $\text{codim}_{\mathbb{R}}$ = # nodes.

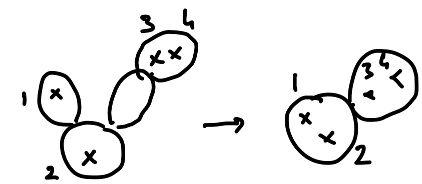
$\mathcal{A}_3 = \{\ \}$ = pt $\qquad \mathcal{A}_4 =$ $\qquad \mathcal{A}_5 =$ $\qquad \mathcal{A}_6$ faces = □ for …, ⬠ for …



- <u>Stable map</u> = domain = nodal Riem. surface, and the map has discrete autom's (i.e. all components of the domain on which the map is constant are stable)

- <u>Gromov-Witten invariants</u>: count genus g curves in given class $\beta \in H_2(M,\mathbb{Z})$ subject to incidence conditions (or domain conditions).

$$[\Sigma,z,u] \in \overline{\mathcal{M}}_{g,k}(M,J,\beta) \xrightarrow{ev} M^k \ni (u(z_1),\dots,u(z_k))$$
$$\downarrow st$$
$\overline{\mathcal{M}}_{g,k} \ni \Sigma/\sim \ \leftarrow$ collapse all unstable components, eg.



If regularity holds (including for bubbled configurations), $\overline{\mathcal{M}}_{g,k}(M,J,\beta)$ smooth compact oriented orbifold of the expected dimension $2d \Rightarrow$ it has a fundamental class $\in H_{2d}(\dots,\mathbb{Q})$ and can $\int$ cohomology class of degree $2d$; do this on classes pulled back by ev & st.

Simplest GW invariants = only evaluation constraints:

$$\alpha_1 \dots \alpha_k \in H^*(M) \rightsquigarrow \langle \alpha_1 \dots \alpha_k \rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,k}(M,J,\beta)]} ev_1^*\alpha_1 \cup \dots \cup ev_k^*\alpha_k .$$

(If α_i Poincaré dual to $[C_i]$: count curves passing through given cycles $C_1,\dots,C_k$)

These are well-defined and indept of auxiliary choices (J, perturb^n scheme, ...) when regularity can be achieved for all configurations in $\overline{\mathcal{M}}_{g,k}(M,J,\beta)$ (or regularization process has been implemented). Now known in full generality; the simplest cases where no advanced techniques are needed =

- when β has the smallest possible energy: $\omega(\beta) = \min\{[\omega]\cdot H_2(M,\mathbb{Z}) \cap \mathbb{R}_+\}$. then no bubbling can occur, nor multiple covers!, so $\mathcal{M}_{g,k}(\beta)$ is already compact and consists of somewhere inj. curves.
- <u>monotone case</u>: $\exists \lambda > 0$ st. $c_1(TM) = \lambda[\omega]$. Then bubbling is controlled by index or more generally, <u>semi-positive case</u>: $A \in \pi_2(M)$, $\omega(A) > 0$, $c_1(TM)\cdot A \geq 3-n \Rightarrow c_1(TM)\cdot A \geq 0$.

Then $\mathcal{M}^*_{0,k}(M,J,\beta) \subset \overline{\mathcal{M}}_{0,k}(M,J,\beta)$ defines a <u>pseudocycle</u> for $J \in \mathcal{J}_{reg}$,

somewhere inj. curves (just one component) — i.e. the complement has $\mathrm{codim}_{\mathbb{R}} \geq 2$ image under evaluation map to M^k.

(This is because:
- stable curves with l nodes are expected to have $\mathrm{codim}_{\mathbb{R}} 2l$ (index comparison)
- if all components are simple and distinct then $J \in \mathcal{J}_{reg}$ gives regularity. dim. is as expected
- if there are multiple covers, look at corresponding simple moduli space (replace all mult. components by underlying simple curve), since all have $c_1(TM)[mA] \geq c_1(TM).[A] \geq 0$ this decreases dim. even further.)

see McDuff-Salamon §6.6.

We can then integrate $\prod ev_i^* \alpha_i$ over $\mathcal{M}^* \subset \overline{\mathcal{M}}$ for $J \in \mathcal{J}_{reg}$ to define GW invts

Outside of semipositive case, need vfc/Kuranishi/polyfolds/stabilizing divisors/...

(small) <u>quantum cohomology</u> over <u>universal Novikov field</u> Λ = completion of group ring of $H_2(M,\mathbb{Z})$:

i.e. $\Lambda = \left\{ \sum_{i=0}^{\infty} a_i q^{\beta_i} \;/\; a_i \in \mathbb{Q} \text{ (or } \mathbb{R}, \mathbb{C}, \dots), \; \beta_i \in H_2(M), \langle [\omega], \beta_i \rangle \to +\infty \right\}$.

$QH^*(M,\Lambda) = H^*(M,\Lambda)$ with "quantum product" $*$, first define $(\alpha_1 * \alpha_2)_\beta$ by

$\forall \beta \in H_2$ (with $\omega(\beta) \geq 0$) $\quad \langle (\alpha_1 * \alpha_2)_\beta \cup \alpha_3, [M] \rangle = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta}$ i.e. cohomology class P.D. to $ev_{3*}(\{$  $\})$

(case $\beta = 0$: constant maps, $(\alpha_1 * \alpha_2)_0 = \alpha_1 \cup \alpha_2$)

(if ev_3 submersion, can define on diff. forms: $(\alpha_1 * \alpha_2)_\beta = ev_{3*}(ev_1^*\alpha_1 \wedge ev_2^*\alpha_2)$ by $\int$ along fibers of ev_3; Stokes + pseudocycle property (no codim. 1 boundary) $\Rightarrow$ Leibniz rule, descends to cohomology).

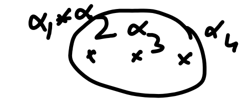
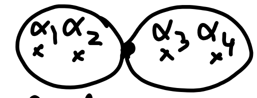
then $\alpha_1 * \alpha_2 := \sum_\beta (\alpha_1 * \alpha_2)_\beta \, q^\beta$. Dim. formula $\Rightarrow \deg((\alpha_1 * \alpha_2)_\beta) = \deg \alpha_1 + \deg \alpha_2 - 2c_1(TM).\beta$

i.e. $*$ graded if we set $\deg(q^\beta) = 2c_1(TM).\beta$.

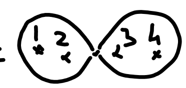
<u>Prop:</u> $*$ is commutative & associative.

<u>Pf:</u> (when regularity holds without perturbations)

<u>commutative</u> = can interchange marked points 1 & 2: $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} = \langle \alpha_2, \alpha_1, \alpha_3 \rangle_{0,\beta}$.

<u>associative</u>: $\langle ((\alpha_1 * \alpha_2) * \alpha_3) \cup \alpha_4, [M] \rangle$ counts  i.e. 

configs representing homology class $\beta = \beta_1 + \beta_2$ count with coefficient $q^{\beta_1} q^{\beta_2} = q^\beta$.

Recall $\overline{\mathcal{M}}_{0,4}(M,J,\beta) \xrightarrow{ev_i} M$, $\downarrow st$, $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1 \ni z_\infty =$ , $z_0 =$ 

Now $\int_{st^{-1}(z) \subset \overline{\mathcal{M}}_{0,4}(J,\beta)} \prod_{i=1}^4 ev_i^* \alpha_i =: \langle \alpha_1 \alpha_2 \alpha_3 \alpha_4 \rangle_{0,\beta,\{z\}}$

(4-point GW invariant with domain constraint) is indep't of $z \in \overline{\mathcal{M}}_{0,4}$ because the cycles $st^{-1}(z)$ are all homologous!

$$\langle ((\alpha_1 * \alpha_2) * \alpha_3) \cup \alpha_4, [M] \rangle = \sum_\beta q^\beta \langle \alpha_1 \alpha_2 \alpha_3 \alpha_4 \rangle_{0,\beta,\{z_\infty\}} = \sum_\beta q^\beta \langle \alpha_1 \alpha_2 \alpha_3 \alpha_4 \rangle_{0,\beta,\{z_0\}} = \langle (\alpha_1 * (\alpha_2 * \alpha_3)) \cup \alpha_4, [M] \rangle.$$